Lecture 1: Intersection Pairings on Smooth Projective Surfaces

Note Title 3/7/2020

X: smooth projective surface

C, D: divisors on X

X smooth $\Longrightarrow$ Weil divisors are Cartier

Goal: define a reasonable intersection pairing $C.D$ "$= \# C\cap D$"

Theorem 1. $\exists!$ $\mathrm{Div}X \times \mathrm{Div}X \longrightarrow \mathbb{Z}$

$C, D, C_1, C_2 \in \mathrm{Div}X$

s.t. (1) C, D nonsingular & intersect transversally

then $C.D = \#(C\cap D)$

notice that complex intersection always contribute positively.

(2) $C.D = D.C$ symmetric

(3) $(C_1+C_2).D = C_1.D + C_2.D$ additive

(4) If $C_1 \sim C_2$, then $C_1.D = C_2.D$

linear equivalence

Lemma 1: $C_1, \cdots, C_r$ irreducible curves, D: very ample on X

$\Rightarrow$ generic $D' \in |D|$ intersect $C_1, \cdots, C_r$ transversally.

pf: This is a direct consequence of Bertini's theorem

Lemma 2: C smooth curves on X

D divisor intersecting C transversally

then $\#(C\cap D) = \underline{\deg(\mathcal{O}_X(D)\otimes \mathcal{O}_C)}$ preserved by linear equivalence pf: $0 \longrightarrow \underbrace{\mathcal{O}_X(-D)\otimes\mathcal{O}_C}_{\substack{\text{invertible sheaf} \\ \text{corresponds to } C\cap D}} \longrightarrow \mathcal{O}_C \longrightarrow \underbrace{\mathcal{O}_{C\cap D}}_{\text{scheme theoretic intersection}} \longrightarrow 0$

the follows from $\deg : \mathrm{Coh}(C) \longrightarrow \mathbb{Z}$, D divisor on C

- $\deg \mathcal{O}_C(D) = \deg D$
- $\mathcal{F}$ torsion free, $\deg \mathcal{F} = \sum_p \mathrm{length}(\mathcal{F}_p)$
- $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

$\Rightarrow \deg \mathcal{F} = \deg \mathcal{F}' + \deg \mathcal{F}''$

Proof of Theorem 1:

Given $C, D \in \mathrm{Div}(X)$, C very ample

Lemma 1 $\Rightarrow \exists C' \in |C|$ s.t $C' \pitchfork D$, C' smooth

Define $C.D := \deg_{C'}(\mathcal{O}_X(D)\otimes\mathcal{O}_{C'}) \underset{\text{Lemma 2}}{=} \#(C'\cap D)$ (1) (4) holds

Notice that if D is also smooth

then $\deg_C(\mathcal{O}_X(D)\otimes\mathcal{O}_C) = \deg_D(\mathcal{O}_X(C)\otimes\mathcal{O}_D)$ (2) holds

For general C, choose H very ample divisor

$\exists n \in \mathbb{N}$ s.t $C+nH$ is very ample

Thus, $H.D$, $(C+nH).D$ are defined $\underset{(3)}{\Longrightarrow}$ $C.D := (C+nH).D - n.(H.D)$

One can similar write $D = (D+nH) - nH$ and play the same trick.

Uniqueness follows from the similar lines above. //

The following is helpful to understand $C.D$ geometrically.

Lemma 3. C, D curves on X w/ no common components

then $C.D = \sum_{p \in \underline{C \cap D}} (C.D)_p$.

where $(C.D)_p = \mathrm{length}\left(\mathcal{O}_{X,P}/(f,g) \right)$

f, g local equation of C, D at P

pf: $0 \to \mathcal{O}_X(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{\underline{C \cap D}} \to 0$

scheme supports on $\underline{C \cap D}$

$\forall p \in \underline{C \cap D}$, its structure sheaf is $\mathcal{O}_{X,P}/(f,g)$

$$\therefore \quad \dim_{\mathbb{C}} H^0(X, \mathcal{O}_{C \cap D}) = \sum_{p \in \underline{C \cap D}} (C.D)_p$$

$\|$ $\mathcal{O}_{C \cap D}$ torsion

$$\chi(\mathcal{O}_{C \cap D}) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_X(-D) \otimes \mathcal{O}_C)$$

$$= \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C) = C.D$$

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We still need to understand self-intersection $C^2 := C.C$

say $|C|$ has base locus C

Assume that C is smooth.

$$C^2 := C.C = \deg_C(\underline{\mathcal{O}_X(C) \otimes \mathcal{O}_C})$$

$\mathcal{J}$: ideal sheaf of C

$$\| $$

$$(\mathcal{O}_X(-C) \otimes \mathcal{O}_C)^\vee = (\mathcal{J}/\mathcal{J}^2)^\vee = \mathcal{N}_{C/X}$$

$$\therefore \quad C^2 = \deg_C(\mathcal{N}_{C/X})$$

Proposition 1 (adjunction formula)

If C nonsingular curve on X of genus g.

then $2g-2 = C.(C+K_X) \geq -2$

pf: $\omega_C \cong (\omega_X \otimes \mathcal{O}_X(C))|_C$ adjunction formula

$\omega_C = K_C$, $\omega_X = K_X$

$2g-2 = \deg \omega_C = \deg(K_X \otimes \mathcal{O}_X(C)|_C) = C.(C+K_X)$

definition of intersection

Remark: If C, D has no common components, then $C.D \geq 0$

In particular, C is numerically effective if $|C|$ has no base points.

Remark: Proposition 1 extends to. C is an effective divisor

w/ $g_a(C) := 1 - \chi(\mathcal{O}_C)$ arithmetic genus

pf: $0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$

$$\begin{aligned} 2-2g_a(C) &= 2\chi(\mathcal{O}_C) \\ &= 2(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C))) \\ &= 2 (-1) \cdot \left(\tfrac{1}{2}(-C)(-C-K) \right) \end{aligned}$$

Riemann-Roch of surfaces